ONSET OF CONVECTION REGIMES WITH DOUBLE PERIOD IN A PERIODIC FIELD OF EXTERNAL FORCES

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We investigate the onset of secondary convective flows in the Rayleigh problem (in a horizontal layer of a viscous incompressible liquid with free boundaries) in the presence of a parameter that varies with time and has a period T, namely the temperature gradient or the intensity of the gravitational field.

The dependence of the critical Rayleigh number on the frequency of the parameter modulation for T-periodic solutions was investigated in [1].

Solutions were considered with double the period and with half the frequency (the so-called half-integer solution). A continuous-fraction algorithm [2, 3] was used to obtain the critical Rayleigh numbers at different values of the modulation frequency. It turned out that, in an appreciable frequency range, it is precisely the solutions with period 2T which are responsible for the appearance of instability, since they correspond to smaller critical values of the Rayleigh number than the T-periodic solutions.

It was established with the aid of the Lyapunov–Schmidt method that at Rayleigh-number values larger than crititical (or close to it) there arises one (accurate to a shift in a horizontal direction) secondary 2T-periodic flow, which is stable against perturbations having the same periodicity and parity as this flow.

1. We consider a flat horizontal layer of an incompressible viscous liquid in a gravitational field that varies periodically with time. The temperature gradient is constant, the mass force is vertical and varies in time in accordance with the law

$$f = g\,(1 + \eta \sin \Omega t), \quad g,\ \eta = \text{const} \tag{1.1}$$

The equations of nonstationary convection under these conditions admit a solution corresponding to relative equilibrium of the liquid in a coordinate system that executes vertical oscillations with acceleration (1.1); the distribution of the temperature is in this case stationary in time and is linear in z.

The dimensionless equations of small perturbations of the equilibrium in the indicated coordinate system are of the form

$$\begin{gathered}\frac{1}{\sqrt{p}}\frac{\partial \Delta\psi}{\partial t} - \Delta\Delta\psi = R\,(1 + \eta \sin \omega t)\frac{\partial \theta}{\partial x} - \frac{1}{p}\left(\frac{\partial \psi}{\partial x}\frac{\partial \Delta\psi}{\partial z} - \frac{\partial \psi}{\partial z}\frac{\partial \Delta\psi}{\partial x}\right)\\ \sqrt{p}\,\frac{\partial \theta}{\partial t} - \Delta\theta = R\,\frac{\partial \psi}{\partial x} - \left(\frac{\partial \psi}{\partial z}\frac{\partial \theta}{\partial x} - \frac{\partial \psi}{\partial x}\frac{\partial \theta}{\partial z}\right)\\ p = \frac{\nu}{\chi}, \quad R^2 = \mathrm{Ra} = \frac{g\beta a h^4}{\nu\chi}, \quad \omega = \frac{\Omega h^2}{\sqrt{\nu\chi}}\end{gathered} \tag{1.2}$$

Here ψ is the stream function, θ is the temperature, p and Ra are the Prandtl and Rayleigh numbers, respectively, and ω is the dimensionless modulation frequency.

Linearizing the system of equations (1.2) in the vicinity of the equilibrium solution, we obtain

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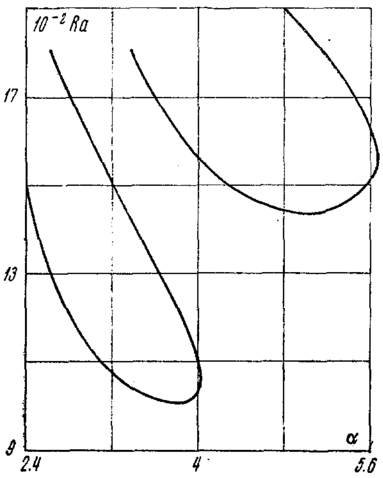


Fig. 1

$$\frac{1}{\sqrt{p}}\frac{\partial\Delta\psi}{\partial t} - \Delta\Delta\psi = R\,(1+\eta\sin\omega t)\frac{\partial\theta}{\partial x} \tag{1.3}$$
$$\sqrt{p}\frac{\partial\theta}{\partial t} - \Delta\theta = R\frac{\partial\psi}{\partial x}$$

We seek the solution of the system (1.3) in the form

$$\psi = \psi(t)\,e^{i(\alpha x+\pi z)}, \qquad \theta = \theta(t)\,e^{i(\alpha x+\pi z)} \tag{1.4}$$

As a result we arrive at one equation* with respect to θ:

$$\theta'' + k^2 q\theta' + \left[k^4 - \frac{\alpha^2}{k^2}\,\mathrm{Ra}\,(1+\eta\sin\omega t)\right]\theta = 0 \tag{1.5}$$
$$q = (1+p)\,p^{-1/2}, \quad k^2 = \alpha^2+\pi^2, \ \theta' = d\theta/dt$$

the solution of which we seek in the form of a series

$$\theta = \sum_{n=-\infty}^{\infty} c_n \exp\,(in\omega t/2) \tag{1.6}$$

It is seen from (1.5) that the coefficients c_n satisfy an infinite system of linear algebraic equations

$$a_n c_n + c_{n-2} - c_{n+2} = 0 \quad (n = 0, \pm 1, \pm 2, \ldots)$$
$$a_n = \frac{2k^4 qn\omega}{\alpha^2\eta\,\mathrm{Ra}} - i\,\frac{2k^4}{\alpha^2\eta\,\mathrm{Ra}}\left(k^4 - \frac{\alpha^2\,\mathrm{Ra}}{k^2} - n^2\omega^2\right) \tag{1.7}$$

We seek for the system (1.7) solutions such that $|c_n| \to 0$ as $|n| \to \infty$. The system (1.7) breaks up into two independent systems of equations for the coefficients c_n with even and odd numbers, respectively. The system for the coefficients with odd numbers corresponds to the previously considered T-periodic solutions [1].

We note that in the case of small amplitudes ($\eta \le 1$) Eq. (1.5) has only T-periodic solutions [5]. This statement is the analog of the so-called "principle of interchange of stability" for stationary problems.

The Fourier coefficients for the 2T-periodic proper solution (which is not T-periodic) is obtained from the system

$$a_{2n+1}c_{2n+1} + c_{2n-1} - c_{2n+3} = 0 \quad (n = 0, \pm 1, \pm 2, \ldots) \tag{1.8}$$

With the aid of the substitution

$$c_{2n+1} = d_n \tag{1.9}$$

we reduce the system to the form

$$a_{2n+1} + \rho_n^{-1} = \rho_{n+1}, \qquad \rho_n = d_n / d_{n-1} \tag{1.10}$$

From this we derive two representations for ρ_n

$$\rho_n = -\cfrac{1}{a_{2n+1} + \cfrac{1}{a_{2n+3} + \ldots}}, \qquad \rho_n = a_{2n-1} + \cfrac{1}{a_{2n-3} + \cfrac{1}{a_{2n-5} \ldots}} \tag{1.11}$$

The condition that these two expressions for ρ_n must coincide leads to a transcendental equation for the determination of Ra

$$a_{-1} + \cfrac{1}{a_{-3} + \cfrac{1}{a_{-5} + \cfrac{1}{a_{-7} + \ldots}}} + \cfrac{1}{a_1 + \cfrac{1}{a_3 + \cfrac{1}{a_5 + \ldots}}} = 0 \tag{1.12}$$

The dependence of the Rayleigh number Ra on the wave number α and on the modulation frequency ω was calculated with the "ODRA-1204" computer.

Figure 1 shows the neutral curve on the (Ra, α) plane. (The remaining parameters have the following values: $\omega = 5$, $\eta = 10$, p = 1.) The instability zone lies above the neutral curve.

*The same equation is obtained in an investigation of the convection in a constant gravitational field, if the temperature gradient is $\nabla\theta_0 = a\,(1+\eta\sin\,\omega t)$ [4].

The most dangerous is the perturbation with the wave number α_* corresponding to the minimum value of Ra_* on the neutral curve. With increasing frequency ω, the critical wave number α_* also increases.

We note that unlike the case of T-periodic solutions, the minimum value of Ra_* is reached at all values of ω on the first lobe.

The critical value of the Rayleigh number also increases with increasing ω. Figures 2a and 2b show the plots of Ra_* and α_* against ω at fixed values $\eta = 10$ and $p = 1$.

A comparison of these plots with the corresponding results of [1] shows that in a considerable range of frequencies the 2T-periodic solutions correspond to smaller Rayleigh numbers, and the T-periodic solutions occur first only at very small or very large frequencies.

2. To investigate 2T-periodic flows that appear after the loss of stability with respect to equilibrium, we use the Lyapunov–Schmidt method [6]. We assume that the critical value of the Rayleigh number Ra_* is simple. We note that the simplicity of Ra_* will subsequently be confirmed by the calculation. The uniqueness of the proper solution (1.6) follows here from the unique solvability of (1.7), and the absence of attached vectors is proved by a numerical verification of the condition of Lemma 1.5 of [3].

We write down the system (1.2) in the form of an operator equation in the space of pairs of functions $w = (\psi, \theta)$ that are periodic in time with a period 2T

$$Aw = RB(t)w + L(w, w)$$

$$Aw = \begin{pmatrix} \frac{1}{\sqrt{p}}\frac{\partial \Delta}{\partial t} - \Delta\Delta & 0 \\ 0 & \sqrt{p}\frac{\partial}{\partial t} - \Delta \end{pmatrix}\begin{pmatrix}\psi \\ 0\end{pmatrix}, \quad Bw = \begin{pmatrix} 0 & \Phi\frac{\partial}{\partial x} \\ \frac{\partial}{\partial x} & 0 \end{pmatrix}\begin{pmatrix}\psi \\ \theta\end{pmatrix} \tag{2.1}$$

$$L(w_1, w_2) = \begin{pmatrix} \frac{1}{p}\left(\frac{\partial \psi_1}{\partial x}\frac{\partial \Delta\psi_2}{\partial z} - \frac{\partial \psi_1}{\partial z}\frac{\partial \Delta\psi_2}{\partial x}\right) \\ \frac{\partial \psi_1}{\partial z}\frac{\partial \theta_2}{\partial x} - \frac{\partial \psi_1}{\partial x}\frac{\partial \theta_2}{\partial z} \end{pmatrix}, \quad \Phi = 1 + \eta\sin\omega t$$

We use $\varphi = (\psi_0, \theta_0)$ to denote the solution of the linearized problem (1.3)

$$Aw = RB(t)w \tag{2.2}$$

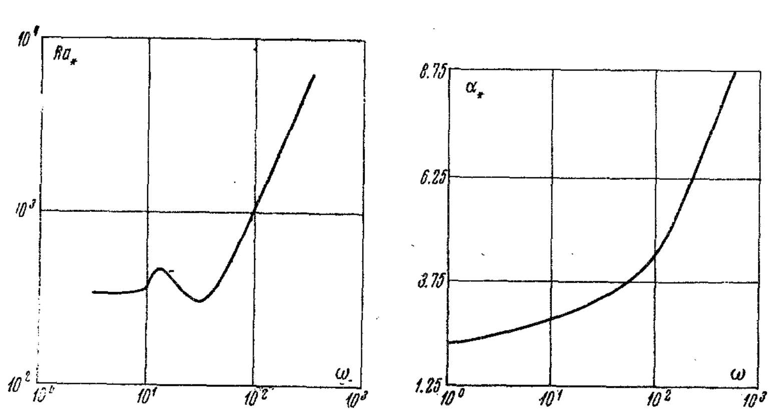
corresponding to the critical value of the parameter $R_* = Ra_*^{1/2}$.

Using the usual Lyapunov–Schmidt scheme, we can deduce that if the constant γ is real and differs from zero, then there exists one nontrivial solution of the problem (1.3) (accurate to within a shift along x [7])

$$w = \gamma\varepsilon\varphi + \gamma^2\varepsilon^2 w_1 + O(\varepsilon^3), \quad \varepsilon = \sqrt{R - R_*} \tag{2.3}$$

The constant γ is determined from the formula

$$\gamma^2 = \left[R_* \int_0^{2T} (L^\circ(\varphi, w_1), w_*)_E \, dt\right]^{-1} \tag{2.4}$$



a b

Fig. 2

Here w_1 is the solution of the inhomogeneous problem

$$Aw - R_* B(t)\, w = L(\varphi, \varphi) \tag{2.5}$$

w_* is the eigenvector of the conjugate equation

$$Aw - R_* B^*(t)\, w = 0, \quad R^* \int_0^{2T} (B\varphi, w_*)_E\, dt = 1 \tag{2.6}$$

The operator L^0 and the scalar product in (2.4) are defined as follows:

$$L^0(u_1, u_2) = L(u_1, u_2) + L(u_2, u_1) \tag{2.7}$$

$$(u_1, u_2)_E = ((\psi_1, \theta_1), (\psi_2, \theta_2))_E = (\psi_1, \psi_2)_{L_2} + (\theta_1, \theta_2)_{L_2}$$

Investigating the variational equation

$$Au - R_* Bu - \varepsilon\gamma L^\circ(\varphi, u) + \varepsilon^2\gamma^2 L^\circ(w_1, u) + \ldots = -\sigma u \tag{2.8}$$

we can show, just as in [8], that the solution (2.3) is stable or unstable, depending on whether the eigenvalue σ_ε of Eq. (2.8), resulting from $\sigma = 0$, is in the left or in the right half-plane.

It is easy to deduce that

$$\sigma_\varepsilon = \zeta(R - R_*)\,\varepsilon^2 + O(\varepsilon^3), \quad \zeta = \left[R_* \int_0^{2T} (\varphi, w_*)_E\, dt\right]^{-1} \tag{2.9}$$

Thus, the flow (2.3) is stable in the linear approximation if $\zeta(R - R_*) > 0$, and is unstable in the opposite case. From the results of [9] it follows that a nonlinear stability also takes place (in a class of perturbations having the same periodicity in x).

The constant γ was calculated in the following manner. We sought the solution of the system (1.3) in the form

$$\psi = \psi_0(t) \sin \alpha x \sin \pi z, \quad \theta = \theta_0(t) \cos \alpha x \sin \pi z \tag{2.10}$$

The amplitude $\theta_0(t)$ was obtained from Eq. (1.5) with the aid of the series (1.6). The coefficients c_{2n+1} $(n = 0, \pm 1, \pm 2, \ldots)$ were determined from the formulas

$$c_1 = 1, \quad c_{2n+1} = \rho_1\rho_2 \ldots \rho_n \quad (n > 0), \quad c_{-2n-1} = \bar{c}_{2n+1}$$

(the bar denotes complex conjugation).

The amplitude $\theta_0(t)$ is now determined from the second equation of the system (1.3). The solutions of the conjugate system are obtained analogously.

The inhomogeneous system (2.5) is in this case

$$\frac{1}{\sqrt{p}} \frac{\partial}{\partial t} \Delta\psi_1 - \Delta\Delta\psi_1 = R_*(1 + \eta \sin \omega t) \frac{\partial\theta_1}{\partial x} \tag{2.11}$$

$$\sqrt{p} \frac{\partial\theta_1}{\partial t} - \Delta\theta_1 = R_* \frac{\partial\psi_1}{\partial x} - \frac{\pi\alpha}{2} \psi_0(t)\, \theta_0(t) \sin 2\pi z$$

We seek the solution of the system (2.11) in the form

$$w_1 = \varphi_1(t) \sin 2\pi z \tag{2.12}$$

The vector function $\varphi_1(t) = (\psi_1(t), \theta_1(t))$ which is periodic with period 2T is obtained from the equations

$$\frac{1}{\sqrt{p}} \psi_1' + 4\pi^2\psi_1 = 0, \quad \sqrt{p}\, \theta_1' + 4\pi^2\theta_1 = -\frac{\pi\alpha}{2} \psi_0\theta_0 \tag{2.13}$$

In this case $\psi_1 = 0$, and θ_1 is easily determined from the known $\psi_0(t)$ and $\theta_0(t)$.

The calculations have shown that $\zeta > 0$ and $\gamma^2 > 0$ for all values of ω.

Thus, at small $Ra - Ra_* > 0$ there exists one stable secondary flow (apart from a shift with respect to x), which is 2T-periodic in time.

We note that, in the case of T-periodic solutions, unstable subcritical regimes set-in in a wide range of frequencies ω. Calculations have shown (see Fig. 2a) that in this frequency range the 2T-periodic regimes correspond to smaller critical values of the Rayleigh number, and on going through these regimes there arise stable 2T-periodic secondary flows.

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